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Numerical Optimization with Differential Equations 1 - WS 2018/2019
Exercise 2

Exercise 1 Consider the parameter dependent initial value problem

$$\dot{x}(t) = f(t, x(t), p), \quad x(t_0) = x_0,$$

where $f \in C^1(\mathbb{R}^{1+n_x+n_p}, \mathbb{R}^{n_x})$ has a globally bounded derivative, $\left\|\frac{\mathrm{d}f}{\mathrm{d}x}(t,x,p)\right\| \le L < \infty$ for all $(t, x, p) \in \mathbb{R}^{1+n_x+n_p}$. For any given (t, t_0, x_0, p), the unique solution is implicitly given by the *Volterra integral equation*

$$F(t, t_0, x_0, p) = x_0 + \int_{t_0}^{t} f(\tau, F(\tau, t_0, x_0, p), p)\mathrm{d}\tau.$$

For any given (t_0, x_0, p) we define the *sensitivity matrices*

$$G^x(t_2, t_1) = \tfrac{\mathrm{d}F}{\mathrm{d}x_1}(t_2, t_1, x_1, p), \quad G^p(t_2, t_1) = \tfrac{\mathrm{d}F}{\mathrm{d}p}(t_2, t_1, x_1, p), \quad x_j := F(t_j, t_0, x_0, p)$$

along the solutions of the initial value problem. Show that the so defined differentiable mappings $F : \mathbb{R}^{2+n_x+n_p} \to \mathbb{R}^{n_x}$ and $G : \mathbb{R}^2 \to \mathbb{R}^{n_x \times n_x}$ satisfy the following properties:

(a) $F(t_0, t_0, x_0, p) = x_0$

(b) $F(t_2, t_0, x_0, p) = F(t_2, t_1, F(t_1, t_0, x_0, p), p) = F(t_2, t_1, x_1, p)$

(c) $\frac{\mathrm{d}F}{\mathrm{d}t_1}(t_1, t_0, x_0, p) = f(t_1, x_1, p)$

(d) $G^x(t_1, t_0) = I + \int_{t_0}^{t_1} \frac{\partial f}{\partial x}(\tau, F(\tau, t_0, x_0, p), p) G^x(\tau, t_0)\mathrm{d}\tau$

(e) $G^p(t_1, t_0) = \int_{t_0}^{t_1} \frac{\partial f}{\partial x}(\tau, F(\tau, t_0, x_0, p), p) G^p(\tau, t_0) + \frac{\partial f}{\partial p}(\tau, F(\tau, t_0, x_0, p), p)\mathrm{d}\tau$

(f) $G^x(t_0, t_0) = I$

(g) $G^x(t_2, t_0) = G^x(t_2, t_1) G^x(t_1, t_0)$

(h) $G^x(t_1, t_0)^{-1} = G^x(t_0, t_1)$

(i) $\frac{\mathrm{d}G^x}{\mathrm{d}t_1}(t_1, t_0) = \frac{\partial f}{\partial x}(t_1, x_1, p) G^x(t_1, t_0)$

(j) $\frac{\mathrm{d}G^x}{\mathrm{d}t_0}(t_1, t_0) = -G^x(t_1, t_0) \frac{\partial f}{\partial x}(t_0, x_0, p)$

(k) $\frac{\mathrm{d}F}{\mathrm{d}t_0}(t_1, t_0, x_0, p) = -G^x(t_1, t_0) f(t_0, x_0, p)$

Hint: The Leibniz-Integral Rule may be helpful, this asserts that for continuously differentiable functions α, β, ϕ the following generalization of the fundamental theorem of calculus holds:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \int_{\alpha(\lambda)}^{\beta(\lambda)} \phi(t, \lambda)\mathrm{d}t = \int_{\alpha(\lambda)}^{\beta(\lambda)} \frac{\partial}{\partial \lambda}\phi(t, \lambda)\mathrm{d}t + f(\beta(\lambda), \lambda)\frac{\partial}{\partial \lambda}\beta(\lambda) - f(\alpha(\lambda), \lambda)\frac{\partial}{\partial \lambda}\alpha(\lambda).$$

(11 Points)

Exercise 2

Specify three illustrative model-examples (realistic scenarios) for the following cases. These three examples are supposed to be taken from different areas per point as mechanics, chemistry, process engineering, economy, aerospace, biology, etc. and should not be from the first sheet.

a) There is a discontinuity in the right-hand side of $f(\cdot)$.

b) There is a discontinuity in the differential states.

c) The inital value x_0 is not fixed, but should be optimized.

d) The end time T of a process is not fixed, but should be optimized.

A description in words is enough. An almost right example for a): The right-hand side determines the growth rate of a (fish-)population. By changing the catch quotas when a threshold value is reached, the catch rate and thus the growth rate also changes.

(4 Points)

M2

Solve the system of differential equation of Sheet 1, Exercise 2 on the interval $[0, 300]$ for $R(0) = 20$, $B(0) = 10$, $\alpha = 0.2$, $\beta = 0.01$, $\gamma = 0.001$ and $\delta = 0.1$ with Matlab. Visualize the solution.

(4 Points)

Hand in solutions on ***Tuesday****, November 6th,* ***at the beginning*** *of the lecture!*
Submit your Matlab solutions until ***Tuesday****, November 6th,* ***11:00 AM*** *by email to your tutor.*