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Numerical Optimization with Differential Equations 1 - WS 2018/2019

Exercise 3

Exercise 1

a) Use the first order necessary optimality conditions to find the extrema of the functions

$$
\begin{aligned}
f(x, y) &= x^2 + y^2 - xy + 2x - 2y \\
g(x, y) &= x^2 + y^2 - 3xy + 2x - 2y \\
h(x, y) &= x^2 + y^2 - 2xy + 2x - 2y.
\end{aligned}
$$

b) Check if the found points are local minima of the functions using the second order conditions.

c) Are the found minima also global minima?

(6 Points)

Exercise 2

Analyze the solvability properties (uniqueness, global, bounded) of the following scalar inital value problems:

a) $\dot{y}(t) = y(t)^2,\ t \geq 0, y(0) = 1,$

b) $\dot{y}(t) = y(t)^{\frac{1}{2}},\ t \geq 0,\ y(0) = 1.$

(4 Points)

Exercise 3

Transfer the forth order system of differential equations

$$
\begin{aligned}
v^{(4)}(t) &= \ddot{v}(t) - 3w(t) \\
w^{(4)}(t) &= 11\dot{v}(t)w(t)
\end{aligned}
$$

into a first order system of differential equations.

(4 Points)

M3

We consider the initial value problem

$$\dot{x}(t) = f(t, x(t), p), \quad x(t_0) = x_0,$$

with $x(t) \in \mathbb{R}^n$ and $p \in \mathbb{R}^m$. The variational differential equations along a solution $x(t)$ are given by

$$
\begin{aligned}
\dot{G}^x(t) &= \frac{\partial f}{\partial x}(t, x(t), p)G^x(t), & G^x(t_0) &= \mathbb{I}_{n\times n}, \\
\dot{G}^p(t) &= \frac{\partial f}{\partial x}(t, x(t), p)G^p(t) + \frac{\partial f}{\partial p}(t, x(t), p), & G^p(t_0) &= 0_{n\times m}.
\end{aligned}
$$

With $G(t) = \begin{pmatrix} G^x(t) & G^p(t) \end{pmatrix} \in \mathbb{R}^{n\times(n+m)}$, they can be written in a single system of the form

$$\dot{G}(t) = \frac{\partial f}{\partial x}(t, x(t), p)G(t) + \begin{pmatrix} 0_{n\times n} & \frac{\partial f}{\partial p}(t, x(t), p) \end{pmatrix}, \quad G(t_0) = \begin{pmatrix} \mathbb{I}_{n\times n} & 0_{m\times n} \end{pmatrix}. \tag{1}$$

If we are only interested in directional derivatives in directions $d^x \in \mathbb{R}^n, d^p \in \mathbb{R}^m$, i.e.,

$$v(t) = G(t) \begin{pmatrix} d^x \\ d^p \end{pmatrix} = G^x(t)d^x + G^p(t)d^p \in \mathbb{R}^n,$$

system (1) can be multiplied from the right hand side by the vector $d = \left(\begin{smallmatrix} d^x \\ d^p \end{smallmatrix}\right) \in \mathbb{R}^{n+m}$ to yield

$$\dot{v}(t) = \frac{\partial f}{\partial x}(t, x(t), p)v(t) + \frac{\partial f}{\partial p}(t, x(t), p)d^p, \quad v(t_0) = d^x. \tag{2}$$

(a) Formulate the variational differential equation (1) with respect to the initial values $x^0 = (R(0), F(0))^T$ and parameters $(\alpha, \beta, \gamma, \delta)$ for the predator-prey system from Sheet 1.

(b) Use `ode45` to solve the combined system

$$\begin{pmatrix} \dot{x}(t) \\ \dot{v}(t) \end{pmatrix} = \begin{pmatrix} f(t, x(t), p) \\ \frac{\partial f}{\partial x}(t, x(t), p)v(t) + \frac{\partial f}{\partial p}(t, x(t), p)d^p \end{pmatrix}, \quad x(t_0) = x^0, v(t_0) = d^x,$$

on the intervall $[0, 300]$ with $x^0 = (20, 10)^T$, $\alpha = 0.2$, $\beta = 0.01$, $\gamma = 0.001$, and $\delta = 0.1$ for each of the six derivative directions

$$d \in \left\{ \begin{pmatrix} 1 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \hline 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \hline 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \hline 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

We are interested in the values of $v(300) = G(300)d$ for each of the six directions of d.

(8 Points)

*Hand in solutions on **Tuesday**, November 13th, **at the beginning** of the lecture!*
*Submit your Matlab solutions until **Tuesday**, November 20th, **11:00 AM** by email to your tutor.*