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Numerical Optimization with Differential Equations 1 - WS 2018/2019

Exercise 5

Exercise 1

Let an $n \times m$ matrix ($n > m$) be given as follows:

$$A = \begin{pmatrix} R & S \\ 0 & 0 \end{pmatrix}$$

with R a regular $p \times p$ triangular matrix (top right $p < n$) and S a $p \times (m-p)$- matrix. Show, using the Penrose axioms:
The Moore-Penrose Inverse $A^\dagger$ can be written as following

$$A^\dagger = \begin{pmatrix} (I_p - VM^{-1}V^T)R^{-1} & 0 \\ M^{-1}V^T R^{-1} & 0 \end{pmatrix}$$

with V and M defined by

$$V := R^{-1}S$$
$$M := I_{m-p} + V^T V.$$

(4 Points)

Exercise 2

We consider the disturbed constrained linear least squares problem

$$\min_{\Delta x} \frac{1}{2}\|F_1 + J_1 \Delta x\|_2^2$$
$$F_2 + J_2 \Delta x = \varepsilon$$

with a small disturbance $\varepsilon \in \mathbb{R}^{n_2}$. (CQ) and (PD) are satisfied. Calculate the solution $\Delta x(\varepsilon)$ and the Lagrange multipliers $\lambda(\varepsilon)$ depending on the disturbance ε.

Let $\lambda = \lambda(0)$ be the Langrange multipliers of the undisturbed problem. Show, that it holds:

$$\frac{d}{d\varepsilon} \frac{1}{2}\|F_1 + J_1 \Delta x\|_2^2 \bigg|_{\varepsilon=0} = \lambda.$$

Hint: Use the implicit function theorem.

(4 Points)

Exercise 3

Use Grönwall's Lemma to prove the "trumpet bound": Let $f : D \subseteq \mathbb{R} \to \mathbb{R}^n \to \mathbb{R}^n$ be continuous and Lipschitz-continuous with respect to the second argument, i.e. there exists $L > 0$ such that

$$\|f(t,y) - f(t,z)\| \leq L\|y - z\| \quad \text{for all } (t,y),(t,z) \in D.$$

Let $y_0, \delta y_0 \in \mathbb{R}^n$ with $\|\delta y_0\| \leq \varepsilon_1$ and $\delta f : [t_0, t_1] \to \mathbb{R}^n$ be a continuous perturbation with $\max_{[t_0,t_1]} \|\delta f(t)\| \leq \varepsilon_2$. Consider the initial value problems

$$\begin{aligned} \dot{y}(t) &= f(t, y(t)), & y(t_0) &= y_0, \\ \dot{z}(t) &= f(t, z(t)) + \delta f(t), & z(t_0) &= y_0 + \delta y_0, \end{aligned}$$

and define $\delta y(t) := z(t) - y(t)$. Then

$$\|\delta y(t)\| \leq (\varepsilon_1 + \varepsilon_2(t - t_0))e^{L(t-t_0)} \quad \text{for all } t \in [t_0, t_1].$$

Grönwall's Lemma: Let

- $u : [t_0, t_1] \to \mathbb{R}$ be continuous
- $a : [t_0, t_1] \to \mathbb{R}$ be continuous and monotonically increasing and
- $b : [t_0, t_1] \to [0, \infty)$ be integrable.

If

$$u(t) \leq a(t) + \int_{t_0}^{t} b(\tau)u(\tau)d\tau \text{ for all } t \in [t_0, t_1],$$

then

$$u(t) \leq a(t) \exp\left(\int_{t_0}^{t} b(\tau)d\tau\right) \text{ for all } t \in [t_0, t_1].$$

(4 Points)

Hand in solutions on ***Tuesday****, November 27th,* ***at the beginning*** *of the lecture!*