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Numerical Optimization with Differential Equations 1 - WS 2018/2019
Exercise 9

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Differential Equations

a. Formulate an initial value problem with ordinary differential equations, including dimensions of variables and functions.
b. How can chemical reactions systems be modeled with ordinary differential equations?
c. Explain the Lotka-Volterra system.
d. Can a higher order differential equation always be transformed to first order?
e. State Peano's theorem.
f. What does local/global Lipschitz continuity mean?
g. State the theorem of Picard–Lindelöf.
h. Give an example of an initial value problem with non-unique solutions.
i. Give an example of an initial value problem that does not admit a solution for all $t > t_0$.
j. State the trumpet bound theorem.
k. What is the integral form of an initial value problem?
l. State the variational differential equations and their initial values.
m. State the ring properties of the solutions of the variational differential equation with respect to initial values.

Discretization methods

a. Explain collocation. (later in the lecture)
b. Explain single and multiple shooting.
c. How do you compute derivatives of the resulting nonlinear least-squares or nonlinear programming problem efficiently?
d. List the advantages and disadvantages of single and multiple shooting.
e. What is the purpose of the condensing algorithm? Which numerical structure does it exploit?

Parameter estimation

a. Formulate a nonlinear unconstrained least squares problem.
b. Formulate a nonlinear equality constrained least squares problem.
c. Formulate a linear equality constrained least squares problem.
d. What do the assumptions [CQ] and [PD] require?

e. State the axioms of the Moore–Penrose pseudoinverse.
f. Which condition must $A \in \mathbb{R}^{m \times n}$ satisfy in order to have a Moore–Penrose pseudoinverse of the form $A^+ = (A^T A)^{-1} A^T$? How can A^+ be computed, if the condition is violated?
g. State the necessary optimality conditions of a linear constrained least squares problem.
h. Prove that $\begin{pmatrix} J_1^T & J_2^T \end{pmatrix}$ has full rank if and only if $J_1^T J_1$ is positive definite on the nullspace of J_2.
i. Show that the necessary optimality conditions of a linear constrained least squares problem admit a unique solution if [CQ] and [PD] hold.
j. What is the difference between the Moore–Penrose pseudoinverse $\begin{pmatrix} J_1 \\ J_2 \end{pmatrix}^+$ and the generalized inverse $\begin{pmatrix} J_1 \\ J_2 \end{pmatrix}^\dagger$?
k. What is an orthonormal matrix?
l. What is a QR decomposition?
m. What is an SVD?
n. Use the QR decomposition to solve a linear least squares problem.
o. Use the SVD to solve a linear least squares problem.
p. How can a linear constrained least squares problem be solved efficiently?
q. State the generalized Gauss–Newton method.
r. State the local contraction theorem.
s. Discuss the two special cases $\kappa = 0$ and $\omega = 0$.
t. Do smaller or larger values of κ appear for least squares problems with large residuals? What does that mean for the speed of convergence of a Gauss–Newton method?

Optimization

a. Formulate a (finite-dimensional) optimization problem with equality and inequality constraints, including dimensions of variables and functions.
b. State the implicit function theorem.
c. State the first and second order necessary optimality conditions.
d. State a sufficient optimality condition.
e. Which conditions imply continuous dependence of the optimal solution on the problem data?
f. Explain Sequential Quadratic Programming.
g. Which matrix must the Hessian in the QP subproblem approximate?

Optimal control (Later in the lecture)

a. Reformulate optimal control problems with free end time to fixed end time.
b. Transform an optimal control problem with Mayer objective to one with Lagrange objective and vice versa.

You do not have to hand in these exercises, there will be a discussion in the tutorial.