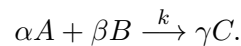


## Numerical Optimization with Differential Equations 1 - WS 2018/2019

### Exercise 11

#### Exercise 1

Let  $\alpha, \beta, \gamma \in \mathbb{N}$  and  $k \in \mathbb{R}$ . A reaction of  $\alpha$  units of substance  $A$  and  $\beta$  units of substance  $B$  into  $\gamma$  units of substance  $C$  can be written as

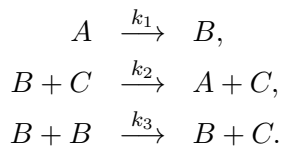


The integers  $\alpha, \beta, \gamma$  are called *stoichiometric factors* and  $k$  is the reaction rate. We model this chemical reaction with a system of ordinary differential equations that describes the temporal change of the concentrations  $[A]$ ,  $[B]$  and  $[C]$  according to

$$\frac{d[A]}{dt} = -\alpha k [A]^\alpha [B]^\beta, \quad \frac{d[B]}{dt} = -\beta k [A]^\alpha [B]^\beta, \quad \frac{d[C]}{dt} = \gamma k [A]^\alpha [B]^\beta.$$

The first equation describes the relation of the reaction rate of  $A$  being proportional to the product of  $[A]^\alpha$  and  $[B]^\beta$ , i.e., the concentrations to the power of their respective stoichiometric factors. Every (elementary) reaction consumes  $\alpha$  units of  $A$ , which leads to the factor  $-\alpha$ . In general, several reactions happen simultaneously and the concentration changes must be added.

Formulate a system of ordinary differential equations for the system of reactions



#### Exercise 2

- a) Use the first order necessary optimality conditions to find the extrema of the functions

$$\begin{aligned} f(x, y) &= x^2 + y^2 - xy + 2x - 2y \\ g(x, y) &= x^2 + y^2 - 3xy + 2x - 2y \\ h(x, y) &= x^2 + y^2 - 2xy + 2x - 2y. \end{aligned}$$

- b) Check if the found points are local minima of the functions using the second order conditions.
- c) Are the found minima also global minima?

### Exercise 3

Transfer the fourth order system of differential equations

$$\begin{aligned}v^{(4)}(t) &= \ddot{v}(t) - 3w(t) \\w^{(4)}(t) &= 11\dot{v}(t)w(t)\end{aligned}$$

in a first order system of differential equations.

### Exercise 4

Let  $J_1 \in \mathbb{R}^{m_1 \times n}$  and  $J_2 \in \mathbb{R}^{m_2 \times n}$  be matrices satisfying

- (CQ)  $\text{Rang}(J_2) = m_2 \leq n$ ,
- (PD)  $\text{Rang} \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} = n \leq m_1 + m_2$ .

Show

- The matrix  $J_1^T J_1 \in \mathbb{R}^{n \times n}$  is positive definite on the nullspace of  $J_2$ , i.e.  $x^T J_1^T J_1 x > 0$  for all  $x \in \mathbb{R}^n \setminus \{0\}$  with  $J_2 x = 0$ .
- The matrix  $\begin{pmatrix} J_1^T J_1 & J_2^T \\ J_2 & 0 \end{pmatrix}$  is regular.

### Exercise 5

In the lecture you got to know the Mayer and the Lagrange cost functional.

- Show, that every Mayer cost functional can be transformed into a Lagrange cost functional.
- Show, that this is also possible in reverse.

### Exercise 6

We consider a vehicle propelled by rockets running on a straight track described by the ODE

$$\dot{s}(t) = v(t), \quad \dot{v}(t) = \frac{u(t)}{m(t)} - c_1 v^2(t), \quad \dot{m}(t) = -c_2 u^2(t),$$

where the states  $s$ ,  $v$ , and  $m$  denote the position, velocity, and mass of the vehicle and the control  $u$  denotes the rocket thrust. The parameters  $c_1$  and  $c_2$  enter in the friction and fuel consumption terms.

- Formulate an optimal control problem to save as much fuel as possible when going from  $s = 0$  to  $s = 10$  within a given time  $T > 0$ . The initial and terminal velocity must be zero. The initial amount of fuel is  $m_0 > 0$ .
- Discretize the problem from (a) with direct multiple shooting and a piecewise constant control discretization on the shooting grid  $0 = t_0 < t_1 < \dots < t_M = T$ . Write down the resulting NLP.

**Exercise 7**

Consider the problem

$$\begin{aligned} \min_{x_1, x_2} \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 - 2 = 0 \end{aligned}$$

Given are  $x^0 = (-1, -1)^T$  and  $\lambda^0 = -1$ .

- Calculate  $x^1 = x^0 + \Delta x^0$  using the full step SQP method with the exact Hessian.
- For this purpose, solve the Quadratic Problem in  $x^0, \lambda^0$ .

**Exercise 8**

Consider for continuously differentiable  $F_1 : \mathbb{R}^n \rightarrow \mathbb{R}^{m_1}$  and  $F_2 : \mathbb{R}^n \rightarrow \mathbb{R}^{m_2}$  the constrained nonlinear least-squares problem

$$\begin{aligned} \min_x \quad & \frac{1}{2} \|F_1(x)\|_2^2 \\ \text{s.t.} \quad & F_2(x) = 0. \end{aligned}$$

Assume that  $x^* \in \mathbb{R}^n$  satisfies  $F_1(x^*) = 0$  and  $F_2(x^*) = 0$  such that [CQ] and [PD] hold in  $x^*$ . Show that for constrained Gauß-Newton and arbitrary  $\varepsilon > 0$  there exists a neighbourhood of  $x^*$  such that the assumptions of the local contraction theorem are satisfied in this neighbourhood with  $\kappa \leq \varepsilon$ .

*You do not have to hand in these exercises, there will be a discussion in the tutorial.*