Nonlinear Programming

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Outline

1. Basic definitions, optimality conditions
2. Algorithms for unconstrained problems
3. Algorithms for constrained problems

Nonlinear Programming Problem

- General problem formulation:

$$
\begin{aligned}
\min f(x) \qquad & f : D \in \mathbb{R}^n \to \mathbb{R} \\
\text{s.t. } g(x) = 0 \qquad & g : D \in \mathbb{R}^n \to \mathbb{R}^m \\
h(x) \geq 0 \qquad & h : D \in \mathbb{R}^n \to \mathbb{R}^k
\end{aligned}
$$

  - x variables
  - f objective function/ cost function/ $\min -f(x) \equiv -\max f(x)$
  - g equality constraints
  - h inequality constraints
  - f, g, h shall be sufficiently smooth (e.g. twice differentiable) functions

Derivatives

- First and second derivatives of the objective function or the constraints play an important role in optimization
- The first order derivatives are called the gradient (of the resp. fct)

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right)^T$$

- and the second order derivatives are called the Hessian matrix

$$\nabla^2 f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \dots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \dots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

Local and Global Solutions

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- Feasible set: $S = \{x \in \mathbb{R}^n : g(x) = 0, h(x) \geq 0\}$

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- x^* global minimizer of f over S $\iff$ $x^* \in S$ and $f(x) \geq f(x^*)$, $\forall x \in S$

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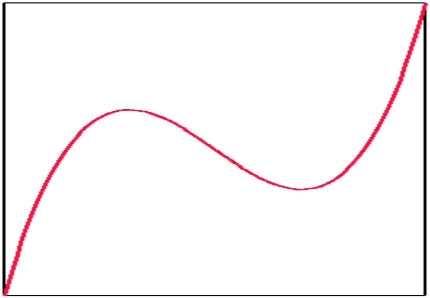
- Feasible set: $S = \{x \in \mathbb{R}^n : g(x) = 0, h(x) \geq 0\}$
- x^* global minimizer of f over $S \iff x^* \in S$ and $f(x) \geq f(x^*)$, $\forall x \in S$
- x^* local minimizer of f over $S \iff x^* \in S$ and there exists $\mathcal{N}(x^*, \delta)$ such that $f(x) \geq f(x^*)$, $\forall x \in S \cap \mathcal{N}(x^*, \delta)$ where $\mathcal{N}(x^*, \delta) := \{x \in \mathbb{R}^n : ||x - x^*||_2 \leq \delta\}$

Local and Global Solutions



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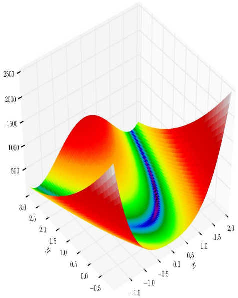

Rosenbrock's test function

see Wikipedia

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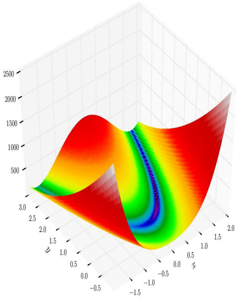

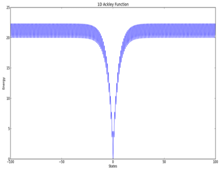

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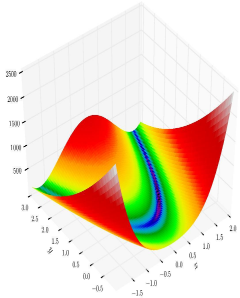

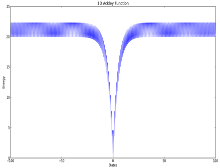

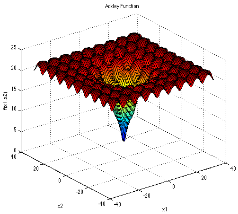

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Main Classes of Continuous Optimization Problems

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- Linear programming: linear objective, linear constraints in the variables

$$\min_{x\in\mathbb{R}^n} \quad c^T x \qquad \text{subject to } a_i^T x = b_i, i \in E, a_i^T x \geq b_i, i \in I,$$

where $c, a_i \in \mathbb{R}^n$, for all i, E and I are finite index sets,

Main Classes of Continuous Optimization Problems

Quadratic programming: quadratic objective, linear constraints in the variables

$$\min_{x\in\mathbb{R}^n} \quad c^T x + \frac{1}{2} x^T H x \quad \text{subject to } a_i^T x = b_i, i \in E, a_i^T x \geq b_i, i \in I,$$

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$$\min_{x \in \mathbb{R}^n} \quad f(x)$$

- Constrained nonlinear programming

$$\min_{x \in \mathbb{R}^n} \quad f(x) \quad \text{subject to } g(x) = 0, h(x) \geq 0.$$

Optimality Conditions for Unconstrained Optimization

$$\min f(x), \quad x \in \mathbb{R}^n$$

- Optimality conditions:
  - give algebraic characteriszations of solutions, suitable for computations
  - provide a way to guarantee that a candidate point is optimal (sufficient conditions)
  - indicate when a point is not optimal (necessary conditions)

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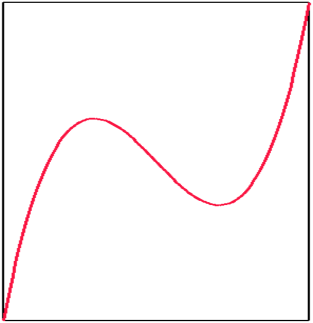
- Necessary conditions:
  x^* is a local minimizer of $f \Rightarrow \nabla f(x^*) = 0$ (stationarity)

Optimality Conditions for Unconstrained Optimization



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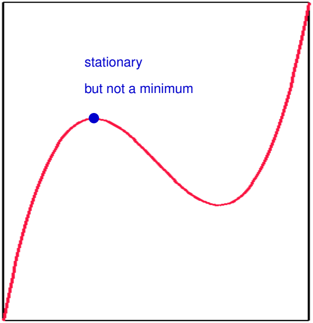

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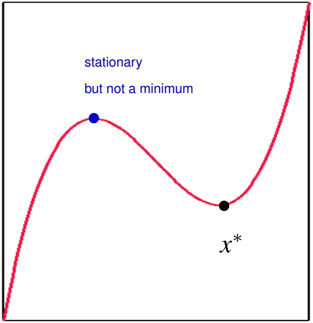
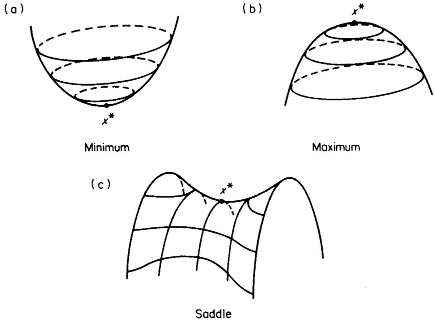


Stationary Points

(a)-(c) x^* is stationary: $\nabla f(x^*) = 0$

(a) $\nabla^2 f(x^*)$ positive definite: local minimum



(b) $\nabla^2 f(x^*)$ negative definite: local maximum

(c) $\nabla^2 f(x^*)$ indefinite: saddle point

Optimality Conditions for Unconstrained Convex Optimization

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  - x^* is stationary $\Rightarrow x^*$ is a global minimizer of f

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  - x^* is a local minimizer of $f \Rightarrow x^*$ is a global minimizer of f
  - x^* is stationary $\Rightarrow x^*$ is a global minimizer of f
- f nonconvex

Optimality Conditions for Unconstrained Convex Optimization

$$\min f(x), \quad x \in \mathbb{R}^n$$

- f convex
  - x^* is a local minimizer of $f \Rightarrow x^*$ is a global minimizer of f
  - x^* is stationary $\Rightarrow x^*$ is a global minimizer of f
- f nonconvex $\rightarrow$ look at higher order derivatives

Second-Order Optimality Conditions for Unconstrained Optimization

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Second-Order Optimality Conditions for Unconstrained Optimization

$$\min f(x), \quad x \in \mathbb{R}^n, \quad f \in \mathcal{C}^2$$

- Necessary second-order conditions:
  x^* is a local minimizer of f $\Rightarrow$ $\nabla^2 f(x^*)$ positive semidefinite
  (f locally convex)

- Sufficient conditions:
  x^* stationary and $\nabla^2 f(x^*)$ positive definite $\Rightarrow$ x^* is a (strict) local minimizer of f

Stability

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Let ε be a perturbation of the problem, then the solution $x(\varepsilon)$ should be a small perturbation of the exact solution x^*:

$$||x(\varepsilon) - x^*|| \leq c||\varepsilon||$$

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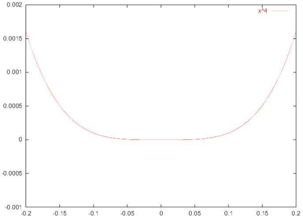
$f(x) = x^4$: Minimum at $x^* = 0$



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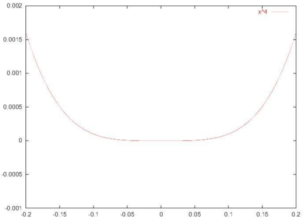
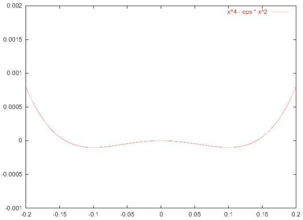


$f(x) = x^4 - \varepsilon x^2$: Maximum at $x = 0$,
Minima at $x = \pm\sqrt{\varepsilon}$

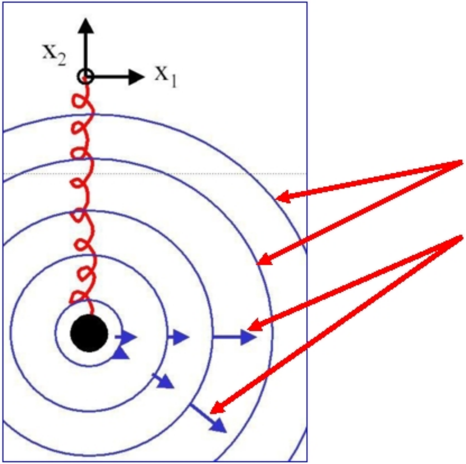


Stability

- In the example problem the sufficient optimality conditions were not satisfied ($\nabla^2 f(x^*)$ is not positive definite)
- One can show:

> Optima that satisfy the sufficient optimality conditions are stable against perturbations

Ball on a spring without constraints



$$\min_{x \in \mathbb{R}^2} \quad x_1^2 + x_2^2 + m x_2$$

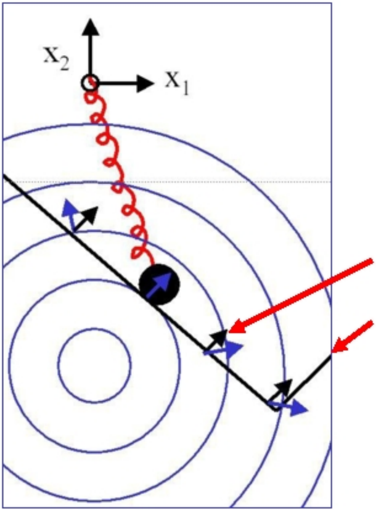
contour lines of $f(x)$

gradient vector
$\nabla f(x) = (2x_1, 2x_2 + m)$

unconstrained minimum:

$$0 = \nabla f(x^*) \Leftrightarrow (x_1^*, x_2^*) = (0, -\frac{m}{2})$$

Ball on a spring with constraints



$$\min_{x \in \mathbb{R}^2} \quad x_1^2 + x_2^2 + m x_2$$

$$h_1(x) = 1 + x_1 + x_2 \geq 0$$

$$h_2(x) = 3 - x_1 + x_2 \geq 0$$

gradient ∇h_1 of active constraint

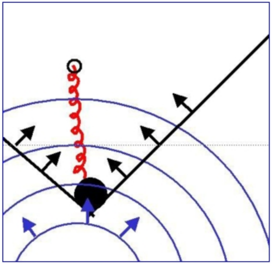
inactive constraint h_2

constrained minimum:

$$\nabla f(x^*) = \mu_1 \nabla h_1(x^*)$$

μ_1 is Lagrange multiplier

Ball on a spring with active constraints



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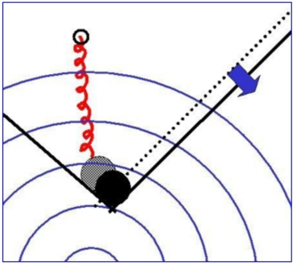
"equilibrium of forces"

$$\nabla f(x^*) \quad = \quad \mu_1 \nabla h_1(x^*) + \mu_2 \nabla h_2(x^*), \quad \mu_1 \geq 0, \mu_2 \geq 0$$

μ_1, μ_2, are Lagrange multipliers

Multipliers as “shadow prices”



old constraint: $h_2(x) \geq 0$

new constraint: $h_2(x) + \varepsilon \geq 0$

- What happens if we relax a constraint?
- Feasible set becomes larger, so new minimum $f(x_\varepsilon^*)$ becomes smaller.
- How much would we gain?

$$f(x_\varepsilon^*) \approx f(x^*) - \varepsilon\mu_2$$

- Multipliers show the hidden cost of constraints.

KKT Conditions for Constrained Optimization

$$\min_{x \in \mathbb{R}^n} f(x), \quad \text{s.t. } g(x) = 0, h(x) \geq 0.$$

KKT Conditions for Constrained Optimization

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- Lagrangian function $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}$

$$\mathcal{L}(x, \lambda, \mu) := f(x) - \sum_i \lambda_i g_i(x) - \sum_i \mu_i h_i(x)$$

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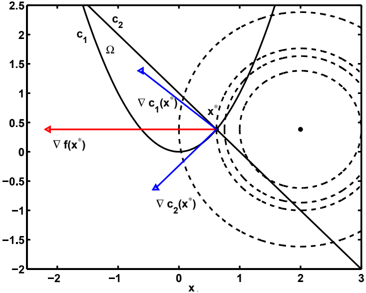
- Karush-Kuhn-Tucker (KKT) point:
  x is a KKT point if there exist $\lambda \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^k$ such that (x, λ, μ) satisfies

$$g(x) = 0, h(x) \geq 0$$

$$\nabla_x \mathcal{L}(x, \lambda, \mu) = 0 \Leftrightarrow \nabla f(x) = \sum_i \lambda_i \nabla g_i(x) + \sum_i \mu_i \nabla h_i(x)$$

$$\mu \geq 0, \mu_i h_i(x) = 0, i = 1, ..., k \Leftrightarrow \mu_i = 0 \text{ or } h_i(x) = 0$$

KKT Conditions: Illustration



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- Examples
  - The Linear Independence Constraint Qualification (LICQ) at $x \Rightarrow$ $\nabla g_i(x), i = 1, ..., m$, $\nabla h_i(x)$, $i \in I(x)$, are linearly independent, where $I(x) := \{i : 1 \leq i \leq k, h_i(x) = 0\}$ “active set”

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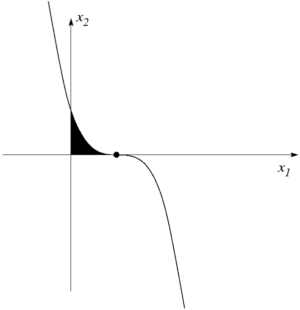
  - All active constraints (equalities and active inequalities) are linear
  - Mangansarian-Fromovitz CQ at $x \Rightarrow \nabla g_i(x), i = 1, ..., m$ linearly independent / or linear and $\exists p \in \mathbb{R}^n$ such that $\nabla g(x)^T p = 0, \ \nabla h_i(x)^T p > 0, \ i \in I(x)$

No CQ: Example from Fiacco, McCormick

$$
\begin{aligned}
h_1(x) &:= (1-x_1)^3 - x_2 &\geq& \quad 0 \quad x_2 \leq (1-x_1)^3 \\
h_2(x) &:= x_1 &\geq& \quad 0 \\
h_3(x) &:= x_2 &\geq& \quad 0
\end{aligned}
$$



$$
\nabla h_1 = \begin{pmatrix} -3(1-x_1) \\ -1 \end{pmatrix}
$$

$$
\nabla h_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
$$

$$
\nabla h_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
$$

Active inequalities in $(1,0)^T$:

$$h_3(x) = 0 \text{ and } h_1(x) = 0$$

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- Second-order necessary optimality conditions:
- Let x^* be optimal and CQ are satisfied in x^*, then x^* is a KKT point and the Hessian $\nabla^2 \mathcal{L}(x^*, \lambda^*, \mu^*)$ of the Lagrange function is positive semidefinite at the tangent set $T(x^*)$:

$$p^T \nabla^2 \mathcal{L}(x^*, \lambda^*, \mu^*) p \geq 0, \forall p \in T(x^*)$$
$$T(x^*) := \{s : s^T \nabla g_i(x^*) = 0,\ i = 1, ..., m,$$
$$s^T \nabla h_i(x^*) \geq 0,\ i \in I(x^*)\}$$

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$$s^T \nabla h_i(x^*) \geq 0,\ i \in I(x^*)\}$$

- Sufficient optimality conditions: if KKT conditions hold and $\nabla^2\mathcal{L}(x^*, \lambda^*, \mu^*)$ is positive definite at $\tilde{T}(x^*, \lambda^*)$, then x^* is optimal, $\tilde{T}(x^*, \lambda^*) := \{s \in T(x^*) : s^T \nabla h_i(x^*) = 0,\ i \in I(x^*)$ with $\mu_i > 0\}$

Algorithms for Unconstrained Optimization

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$$\min f(x) \qquad x \in \mathbb{R}^n$$

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$$\min f(x) \qquad x \in \mathbb{R}^n$$

- Find a local minimizer x^* of $f(x)$, i.e. a point satisfying
  - $\nabla f(x^*) = 0$ (stationarity)
  - and $\nabla^2 f(x^*)$ positive definite

Algorithms for Unconstrained Optimization

- Basic structure of most algorithms:
  - choose start value x^0
  - for $k = 1, \ldots,$
    - determine search (descent) direction p^k
    - determine steplenght α^k
    - new iterate $x^{k+1} = x^k + \alpha^k p^k$
    - check for convergence
- Optimization algorithms differ in the choice of p^k and α^k

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- Two types of convergence:

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  - Local convergence: convergence of the full-step ($\alpha_k \equiv 1$) algorithm near the solution
  - Global convergence: convergence of an algorithm starting from an any arbitrary point x^0

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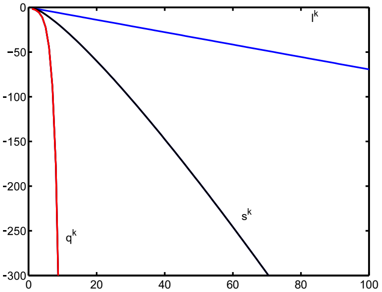
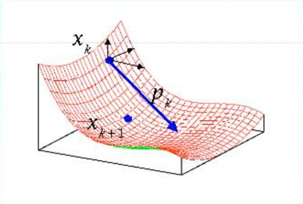
Rates of Convergence



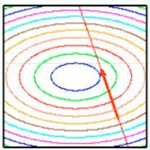
Generic Linesearch Algorithm

Search direction p^k:
f must decrease
along the direction p^k
$\nabla f(x^k)p^k < 0$



Steplength α^k
to guarantee global convergence:
solve $1D$ minimization problem
(exact or inexact):

$$\alpha^k = \text{arg}\min_{\alpha} f(x^k + \alpha p^k)$$



Computation of Steplength

- Ideal: Move to (global) minimum on the selected line (univariate optimization, exact line search)

$$\alpha^k = \text{arg} \min_{\alpha} f(x^k + \alpha p^k)$$

- In practice: approximate solution may guarantee global convergence, perform only inexact line search

$$\alpha^k \approx \text{arg} \min_{\alpha} f(x^k + \alpha p^k)$$

- Problem: how to guarantee sufficient decrease?
  - Answer: Check e.g. if α^k satisfies Armijo-Wolfe conditions

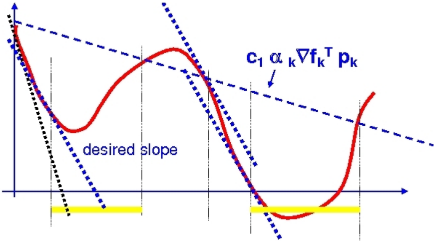
Armijo-Wolfe Conditions for Inexact Line Search

- Armijo condition (sufficent decrease condition):

$$f(x^k + \alpha p^k) \leq f(x^k) + c_1 \alpha^k \nabla^T f(x^k) p^k, \quad c_1 \in (0, 1)$$

- Curvature condition:

$$\nabla^T f(x^k + \alpha p^k) p^k \geq c_2 \nabla^T f(x^k) p^k, \quad c_2 \in (c_1, 1)$$



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  - or $\min\{\frac{|\nabla f(x^k)^T p^k|}{||s^k||}, |\nabla f(x^k)^T p^k|\} \to 0$ as $k \to \infty$

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  - not only p^k to be descent direction
  - but also $\cos \theta^k \geq \delta > 0$ for all k
    (i.e.p^k and $\nabla f(x^k)$) should not become nearly orthogonal!

Computation of the Search Direction

- For the determination of p^k frequently first and second order derivatives of $f(x^k)$ are used
- We discuss:
  - Steepest descent method
  - Newton's method
  - Quasi-Newton methods
- Left out: conjugate gradients

Algorithm 1: Steepest Descent Method

- Based on first order Taylor series approximation of objective function

$$f(x^k + p^k) = f(x^k) + \underbrace{\nabla^T f(x^k) p^k}_{} + \ldots$$

- maximum descent, if

$$\frac{\nabla^T f(x^k) p^k}{||p^k||} \to \min!$$

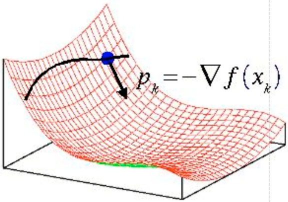
$$\Rightarrow p^k = -\nabla f(x^k)$$

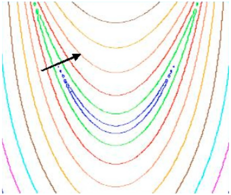
Algorithm 1: Steepest Descent Method

- Choose steepest descent direction, perform (exact) line search:

$$p^k = -\nabla f(x^k) \quad x^{k+1} = x^k - \alpha^k \nabla f(x^k)$$

- search direction is perpendicular to level sets of $f(x)$





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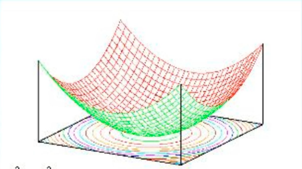
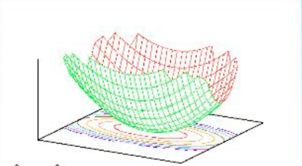
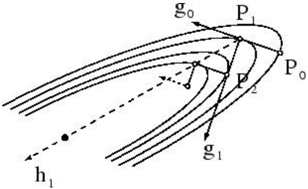


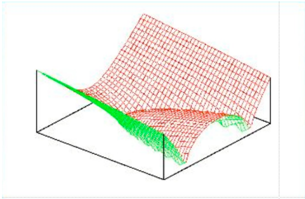
$x^2+y^2 \rightarrow C=0$



$x^2+5y^2 \rightarrow C\approx 0.6$



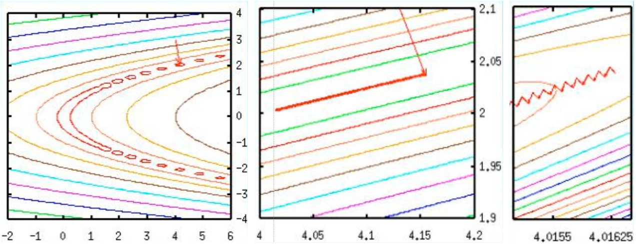
Example - Steepest Descent Method



$$f(x,y) = \left((x-y^2)^2 + \frac{1}{100} \right)^{\frac{1}{4}} + \frac{1}{100} y^2$$

banana valley function

global minimum at $x = y = 0$



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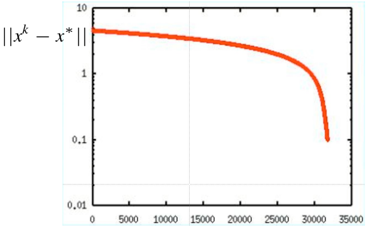

Convergence of steepest descent method:

- needs almost 35.000 iterations to come closer than 0.1 to the solution
- mean value of convergence rate $C \approx 0.99995$
- it holds at $(x = 4, y = 2)$

$$\lambda_{min} = 0.1, \lambda_{max} = 268, C \approx \frac{268 - 0.1}{268 - 0.1} \approx 0.9993$$

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- global convergence under weak assumptions, but no second-order optimality guarantees for the generated solution
- scale-dependent: when the objective poorly scaled, very slow convergence, cumulation of round-off errors and break-down
- useful for some special applications (e.g. in data analysis)

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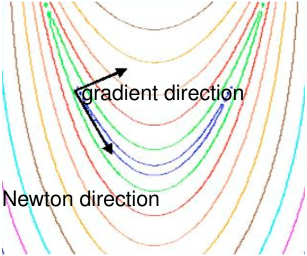
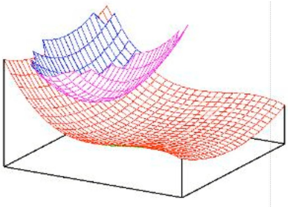
- p^k is decrease direction if the hessian $\nabla^2 f(x^k)$ is positive definite!

Visualization of Newton's method

- p^k minimizes quadratic approximation of the objective

$$Q(p^k) = f(x^k) + \nabla^T f(x^k) p^k + \frac{1}{2} (p^k)^T \nabla^2 f(x^k) p^k$$





Why is it called Newton's method?

- Newton's method finds zeros of nonlinear equations. Here: find solution of the equation

$$\nabla f(x) = 0$$

- Newton's idea: use Taylor series of ∇f at x^k:

$$\nabla f(x^k + p^k) \approx \nabla f(x^k) + \nabla^2 f(x^k) p^k = 0!$$

- and to make this zero, set:

$$p^k = \underbrace{-(\nabla^2 f(x^k))^{-1} \nabla f(x^k)}_{\textit{Newton direction}}$$

- (Full step) Newton's method: iterate

$$x^{k+1} = x^k + p^k$$

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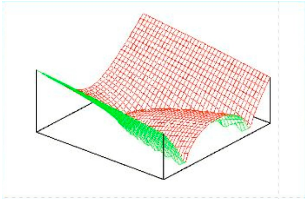
  - Ensure this by: Levenberg-Marquardt, or trust-region methods

Example - Newton's method



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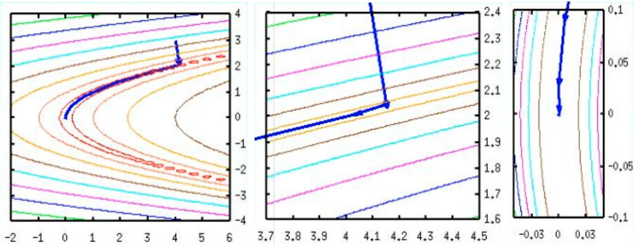

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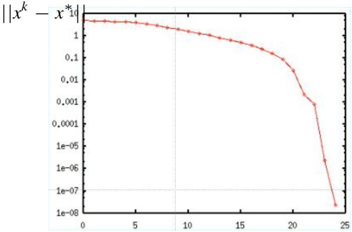
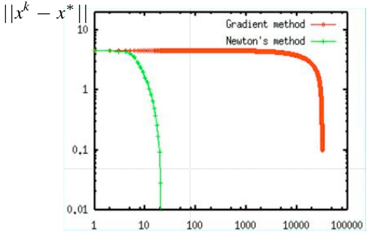


- Convergence of Newton's method:
  - less than 25 iterations for an accuracy of better than 10^{-7}!
  - convergence roughly linear for first 15-20 iterations since step length $\alpha^k \neq 1$
  - convergence roughly quadratic for last iterations with step length $\alpha^k = 1$

Comparison of the Steepest Descent and Newton's Methods



- For banana valley example:
  - Newton's method much faster than steepest descent method (factor 1000)
  - Newton's method superior due to higher order of convergence
  - steepest descent method converges too slowly for practical applications

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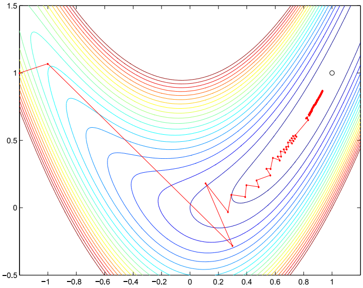

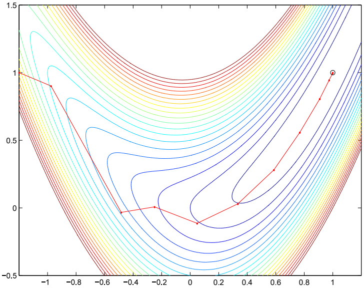

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- Idea: approximate Hessian matrix $\nabla^2 f(x^k)$
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- special case: steepest descent method: $B = I$

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  - or: use same matrix B for several iterations:

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Quasi-Newton Methods

- different Quasi-Newton methods:
  - simplified Newton method: keep Hessian approximation B constant, e.g.

$$B^k \equiv \nabla^2 f(x^0)$$

  - or: use same matrix B for several iterations:

$$\text{if } \frac{||\Delta x^{k+1}||}{\Delta x^k||} \geq \delta_{max} \text{ then update } B^k \equiv \nabla^2 f(x^k)$$

  - or, even cheaper: use update-formulas for Hessian...

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- Idea: Given an Hessian approximation B^k
- find a new approximation B^{k+1} that is "close" to B^k and satisfies

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- Examples:
  - Symmetric Broyden-update
  - DFP-update (Davidon, Fletcher, Powell)
  - BFGS-update (Broyden, Fletcher, Goldfarb, Shanno) (most widely used)

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- Quasi-Newton methods most popular method for medium scale problems

Constrained Optimization: SQP-method

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- Constrained problem:

$$
\begin{aligned}
\min f(x) \qquad & f : D \in \mathbb{R}^n \to \mathbb{R} \\
\text{s.t. } g(x) = 0 \qquad & g : D \in \mathbb{R}^n \to \mathbb{R}^l \\
h(x) \geq 0 \qquad & h : D \in \mathbb{R}^n \to \mathbb{R}^k
\end{aligned}
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- Idea: Consider successively quadratic approximations of the problem:

$$\begin{aligned} \min_p \qquad & f(x^k) + \nabla^T f(x^k) p + \frac{1}{2} p^T H^k p \\ \text{s.t.} \qquad & g(x^k) + \nabla g(x^k) p = 0 \\ & h(x^k) + \nabla h(x^k) p \geq 0 \end{aligned}$$

- $H^k \approx \nabla^2 L(x, \lambda, \mu)$

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- global convergence can be achieved by using a stepsize strategy based on (inexact) 1D minimization of an appropriate merit function, e.g. exact merit function

$$T(x) = f(x) + \sum_{eq} \gamma_i |g_i(x)| + \sum_{ineq} \beta_i |\min\{0, h_i(x)\}|$$

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  with sufficiently large γ_i, β_i
- alternatively, global convergence by trust region

Constrained Optimization: SQP-method

1. Start with $k = 0$, start value x^0 and $H^0 = I$
2. Compute $f(x^k), g(x^k), h(x^k), \nabla f(x^k), \nabla g(x^k), \nabla h(x^k)$
3. If x^k feasible and $||\nabla \mathcal{L}(x, \lambda, \mu)|| < \varepsilon$ then stop $\rightarrow$ convergence achieved
4. Solve quadratic problem (QP) and get p^k
5. Perform line search and get stepsize α^k
6. Iterate $x^{k+1} = x^k + \alpha^k p^k$
7. Update Hessian of the Lagrange function
8. $k = k + 1$, goto step 2

Solution of the Quadratic Program

- Unconstrained case:

$$\min_p g^T p + \frac{1}{2} p^T H p$$

- H must be positive definite, otherwise the optimization problem has no solution
- necessary optimality condition:

$$Hp + g = 0$$

- $=>$ use cholesky-method or cg-method to solve

Solution of the Quadratic Program

- equality constrained case:

$$\min_p g^T p + \frac{1}{2} p^T H p$$
$$Ap + a = 0$$

- necessary optimality condition (KKT-system): $\exists \lambda$ such that

$$\begin{pmatrix} H & A^T \\ A & 0 \end{pmatrix} = - \begin{pmatrix} g \\ a \end{pmatrix}$$

- matrix is indefinite, use range- or nullspace-method to solve

Solution of the Quadratic Program

- equality and inequality constrained case:

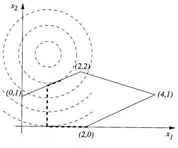
$$\min_p g^T p + \frac{1}{2} p^T H p$$
$$Ap + a = 0$$
$$Bp + b \geq 0$$

- use active-set-strategy
- aim: find out which inequalities are active at the solution and which not
- idea: solve a sequence of equality constrained QPs

$$\begin{aligned} \min_p \quad & g^T p + \frac{1}{2} p^T H p \\ & Ap + a = 0 \\ & B_i p + b_i = 0, \quad i \in W^k \end{aligned}$$

where W^k is a “guess” for an optimal active set

Active-Set Strategy: Example



- $p^0 = (2,0)^T$, $W^0 = \{3,5\}$, negative multiplier with respect to constraint 3, remove constraint 3

$$
\begin{aligned}
\min (p_1 - 1)^2 + (p_2 - 2.5)^2 \\
p_1 + 2p_2 + 2 &\geq 0 \\
-p_1 - 2p_2 + 6 &\geq 0 \\
-p_1 - 2p_2 + 2 &\geq 0 \\
p_1 &\geq 0 \\
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\end{aligned}
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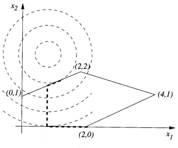

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- $p^1 = (2,0)$, $W^1 = \{5\}$, no negative multipliers, solve QP, step length $\theta = 1$

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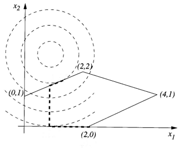

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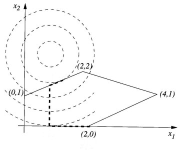
- $p^1 = (2,0)$, $W^1 = \{5\}$, no negative multipliers, solve QP, step length $\theta = 1$
- $p^2 = (1,0)$, $W^2 = \{5\}$, negative multiplier respect to constraint 5, remove constraint 5

Active-Set Strategy: Example



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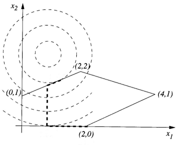
- $p^3 = (1,0)$, $W^3 = \{\}$, no negative multipliers, solve QP, step length $\theta < 1$, constraint 1 gets active

Active-Set Strategy: Example



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- $p^4 = (1,1.5)$, $W^4 = \{1\}$, no negative multipliers, solve QP, step length $\theta = 1$

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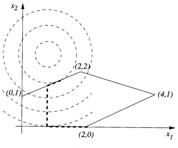

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- $p^3 = (1,0)$, $W^3 = \{\}$, no negative multipliers, solve QP, step length $\theta < 1$, constraint 1 gets active
- $p^4 = (1,1.5)$, $W^4 = \{1\}$, no negative multipliers, solve QP, step length $\theta = 1$
- $p^5 = (1.4,1.7)$, $W^5 = \{3,5\}$ all multipliers positive $\rightarrow$ solution

Alternative Methods for Constrained Optimization

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- Penalty and barrier methods

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- Augmented lagrangian methods

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- Augmented lagrangian methods
- Interior point methods for inequality constrained problems

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- Starting at an initial guess x^0 , most methods iterate $x^{k+1} = x^k + \alpha^k p^k$ with search direction p^k and step length α^k

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- Search direction can be chosen differently
  - steepest descent (simple, but slow and rarely used in practice)
  - Newton's method (very fast if Hessian cheaply available)
  - Quasi-Newton methods (cheap, fast, and popular, e.g. BFGS)
  - SQP methods for constrained optimization
  - CG method (good for very large scale problems)

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  - CG method (good for very large scale problems)
- Other methods: direct search, simulated annealing, genetic algorithms, ... useful for special optimization problems

References

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Thank you for your attention!